

STRUCTURE AND REPRESENTATIONS OF THE QUANTUM GENERAL LINEAR SUPERGROUP

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Abstract

The structure and representations of the quantum general linear supergroup $GL_q(m|n)$ are studied systematically by investigating the Hopf superalgebra G_q of its representative functions. G_q is factorized into $G_q^\pi G_q^{\bar{\pi}}$, and a Peter - Weyl basis is constructed for each factor. Parabolic induction for the quantum supergroup is developed. The underlying geometry of induced representations is discussed, and an analog of Frobenius reciprocity is obtained. A quantum Borel - Weil theorem is proven for the covariant and contravariant tensorial irreps, and explicit realizations are given for classes of tensorial irreps in terms of sections of quantum super vector bundles over quantum projective superspaces.

1 INTRODUCTION

Quantized universal enveloping superalgebras [1] [2] (which will be called quantum superalgebras for simplicity) represent the most important generalizations of the Drinfeld - Jimbo [3] quantized universal enveloping algebras. Their origin can be traced back to the Perk - Schultz solution of the Yang - Baxter equation and also the work of Bazhanov and Shadrnikov[4]. However, systematical investigations of such algebraic structures only started about six years ago, but in an intensive manner. By now the subject has been developed quite extensively: the quasi - triangular Hopf superalgebraic structure of the quantum superalgebras was investigated [5]; the representation theory of large classes of quantum (affine)superalgebras and super Yangians was developed [6][7]; applications of quantum superalgebras to integrable two dimensional models in statistical mechanics and quantum field theory were extensively explored [1][8]. Quantum superalgebras have also been applied to the study of knot theory and 3 - manifolds [9][10], yielding many new topological invariants, notably, the multi - parameter generalizations of Alexander - Conway polynomials.

Closely related to the Drinfeld - Jimbo algebras are the quantum groups introduced by Woronowicz and Faddeev - Reshetikhin - Takhtajan [11], which are, in the spirit of Tannaka - Krein duality theory, the ‘groups’ associated with the quantized universal enveloping algebras. One very important aspect of quantum groups is their geometrical significance: they provide a concrete framework for developing noncommutative

geometry [12], in particular, for investigating notions such as quantum flag varieties [13] and quantum fibre bundles.

Our aim here is to study the structure and representations of the quantum general linear supergroup $GL_q(m|n)$ in a systematical fashion by investigating the algebra of its representative functions. We start in section 2 with a concise treatment of finite dimensional unitary representations of $U_q(gl(m|n))$. Results will be repeatedly used in the remainder of the paper. In section 3 we define the quantum general linear supergroup $GL_q(m|n)$, or more exactly, the superalgebra G_q of functions on it. This is done by first defining the bi - superalgebras G_q^π and $G_q^{\bar{\pi}}$, which are respectively generated by the matrix elements of the vector irrep and its dual irrep. Peter - Weyl type of bases for these bi - superalgebras are constructed. The G_q is defined to be generated by G_q^π and $G_q^{\bar{\pi}}$ with some extra relations. It has the structures of a $*$ - Hopf superalgebra, which separates points of $U_q(gl(m|n))$, and factorizes into $G_q^\pi G_q^{\bar{\pi}}$. Section 4 treats the representation theory of the quantum supergroup, and in particular, parabolic induction. The geometrical interpretation of induced representations is discussed, leading naturally to the concepts of quantum homogeneous spaces and quantum super vector bundles. A quantum analog of Frobenius reciprocity is obtained; and a quantum version of the Borel - Weil theorem is proven for the covariant and contravariant tensorial irreps. Section 5 gives the explicit realizations of two infinite classes of tensorial irreps in terms of sections of quantum super vector bundles over the quantum projective superspace. In doing this, we also treat the quantum projective superspace in some detail.

2 UNITARITY REPRESENTATIONS OF $U_q(gl(m|n))$

The finite dimensional unitary representations of $U_q(gl(m|n))$ were classified in [15]. Here we will reformulate the results on the covariant and contravariant tensor irreps so that they can be readily used in the remainder of the paper. The material presented here also heavily relies on references [6] and [14].

2.1 Hopf $*$ - superalgebras and unitary representations

Let A be a \mathbb{Z}_2 - graded associative algebra over the complex field \mathbb{C} . Its underlying \mathbb{Z}_2 - graded vector space is the direct sum $A = A_0 \oplus A_1$ of the even subspace A_0 and the odd subspace A_1 . We introduce the grading index $[\] : A_0 \cup A_1 \rightarrow \mathbb{Z}_2$ such that $[a] = \theta$ if $a \in A_\theta$. We will call A a \mathbb{Z}_2 - graded $*$ - algebra, or $*$ - superalgebra, if there exists an even anti - linear anti - automorphism $*$: $A \rightarrow A$ such that $*$ \circ $*$ = id_A . We will denote $*(a)$ by a^* . Needless to say, $*(ab) = b^*a^*$, $a, b \in A$.

An important new feature of the \mathbb{Z}_2 - graded case is that for a given $*$ - operation of A , there exists an associated $*'$ such that

$$*'(a) = (-1)^{[a]}a^*, \quad (1)$$

for a being homogeneous, and extends to the whole of A anti - linearly. There also exist the so called graded $*$ - operations, which, however, are not useful for this paper, thus will not be discussed any further.

Let A and B be two \mathbb{Z}_2 - graded $*$ - algebras. Then $A \otimes_{\mathbb{C}} B$ has a natural \mathbb{Z}_2 - graded $*$ - algebra structure, with the $*$ - operation defined for homogeneous elements

by

$$*(a \otimes b) = (-1)^{[a][b]} a^* \otimes b^*,$$

and for all the elements by extending this anti - linearly.

Consider a \mathbb{Z}_2 - graded Hopf algebra (also called Hopf superalgebra) H , with multiplication m , unit 1_H , co - multiplication Δ , co - unit ϵ and antipode S . We emphasize that the antipode is a *linear* anti- automorphism of the underlying algebra of H . In particular, for homogeneous $a, b \in A$, we have $S(ab) = (-1)^{[a][b]} S(b)S(a)$. H will be called a \mathbb{Z}_2 - graded Hopf $*$ - algebra, or Hopf $*$ - superalgebra, if the underlying algebra of H is a $*$ - superalgebra such that Δ and ϵ are $*$ - homomorphisms, i.e.,

$$* \circ \Delta = \Delta \circ *, \quad * \circ \epsilon = \epsilon \circ *.$$

These properties together with the defining relations of the antipode

$$m \circ (S \otimes id) \Delta = m \circ (id \otimes S) \Delta = 1_H \epsilon$$

imply that

$$S \circ * \circ S \circ * = id_H.$$

Let V be a left H - module. If there exists a non - degenerate sequilinear form $(\ , \) : V \otimes V \rightarrow \mathbb{C}$, such that

$$\begin{aligned} (i). \quad & (av, u) = (v, a^*u), \quad \forall u, v \in V, a \in H, \\ (ii). \quad & (v, v) \geq 0, \quad (v, v) = 0 \text{ iff } v = 0, \end{aligned}$$

we call V and the associated representation of H unitary.

Unitary representations have the following important properties

- i). *A unitary representation is completely reducible;*
- ii). *The tensor product of two unitary (with respect to the same $*$ - operation) representations is again unitary;*
- iii). *If a representation is unitary with respect to $*$, then its dual is unitary with respect to $*'$.*

All the three assertions are well known, but there are some related matters worth discussing. One is concerned with the requirement that two representations must be unitary with respect to the same $*$ - operation in order for their tensor product to be unitary as well. The tensor product $V \otimes_{\mathbb{C}} W$ of two H - modules has a natural H module structure

$$a\{v \otimes w\} = \sum_{(a)} (-1)^{[a_{(2)}][v]} a_{(1)}v \otimes a_{(2)}w.$$

If both V and W are equipped with sequilinear forms $(\ , \) : V \otimes_{\mathbb{C}} V \rightarrow \mathbb{C}$, and $(\ , \) : W \otimes_{\mathbb{C}} W \rightarrow \mathbb{C}$, we can define a sequilinear form $((\ , \)) : (V \otimes_{\mathbb{C}} W)^{\otimes 2} \rightarrow \mathbb{C}$ by

$$((v_1 \otimes w_1, v_2 \otimes w_2)) = (v_1, v_2)(w_1, w_2).$$

Now if both V and W are unitary with respect to the same $*$ - operation, then $((,))$ is clearly positive definite and nondegenerate. Furthermore,

$$\begin{aligned} ((v_1 \otimes w_1, a\{v_2 \otimes w_2\})) &= \sum_{(a)} (-1)^{[a_{(2)}][v_2]} ((a_{(1)}^* v_1 \otimes a_{(2)}^* w_1, v_2 \otimes w_2)) \\ &= ((a^* \{v_1 \otimes w_1\}, v_2 \otimes w_2)). \end{aligned}$$

Therefore, $V \otimes_{\mathbb{C}} W$ indeed furnishes a unitary H - module. On the other hand, if, say, V is $*$ - unitary, while W is $*'$ - unitary, then one can easily see that the above calculations will fail to go through.

The other concerns the third assertion, the validity of which actually requires some qualification, namely, the Hopf $*$ - superalgebra H in question must admit an even group like element $K_{2\rho}$ satisfying

$$K_{2\rho}^* = K_{2\rho}, \quad S^2(a) = K_{2\rho} a K_{2\rho}^{-1}, \quad \forall a \in H. \quad (2)$$

Let V be a locally finite module over H , which is unitary with respect to the sequilinear form $((,)) : V \otimes V \rightarrow \mathbb{C}$. For every $v \in V$, we define v^\dagger by $v^\dagger(w) = (v, w)$, $\forall w \in V$, and denote the linear span of all such v^\dagger by V^\dagger , which is a subspace of the dual vector space of V . The V^\dagger has a natural H module structure, with the action of H given by

$$(av^\dagger)(w) = (-1)^{[a][v^\dagger]} v^\dagger(S(a)w), \quad w \in V.$$

Unitarity of V leads to

$$av^\dagger = (-1)^{[a][v]} (*S(a)v)^\dagger.$$

We define a sequilinear form $((,))' : V^\dagger \otimes V^\dagger \rightarrow \mathbb{C}$ by

$$(v^\dagger, w^\dagger)' = (K_{2\rho} w, v).$$

It follows from the properties of the original form on V that $((,))'$ is positive definite and nondegenerate. A straightforward calculation shows that

$$(av^\dagger, w^\dagger)' = (v^\dagger, *'(a)w^\dagger)',$$

where $*'$ is defined by (1).

2.2 $U_q(gl(m|n))$

Throughout the paper, we will denote by \mathfrak{g} the complex Lie superalgebra $gl(m|n)$, and by $U(\mathfrak{g})$ its universal enveloping algebra. As is well known, there are the Drinfeld and Jimbo two versions of the quantized universal enveloping algebra $U_q(\mathfrak{g})$ of \mathfrak{g} , which, though, have very similar properties at generic q .

It is the Jimbo version of $U_q(\mathfrak{g})$ that will be used in this paper. Now $U_q(\mathfrak{g})$ is a \mathbb{Z}_2 - graded unital associative algebra over $\mathbb{C}(q, q^{-1})$, q being an indeterminate, generated by $\{K_a, K_a^{-1}, a \in \mathbf{I}; E_b, b \in \mathbf{I}\}$, $\mathbf{I} = \{1, 2, \dots, m+n\}$, $\mathbf{I}' = \{1, 2, \dots, m+n-1\}$,

subject to the following relations

$$\begin{aligned}
K_a K_a^{-1} &= 1, & K_a^{\pm 1} K_b^{\pm 1} &= K_b^{\pm 1} K_a^{\pm 1}, \\
K_a E_{b \pm 1} K_a^{-1} &= q_a^{\delta_{ab} - \delta_{a \pm 1}} E_{b \pm 1}, \\
[E_{a a+1}, E_{b+1 b}] &= \delta_{ab} (K_a K_{a+1}^{-1} - K_a^{-1} K_{a+1}) / (q_a - q_a^{-1}), \\
(E_{m m+1})^2 &= (E_{m+1 m})^2 = 0, \\
E_{a a+1} E_{b b+1} &= E_{b b+1} E_{a a+1}, \\
E_{a+1 a} E_{b+1 b} &= E_{b+1 b} E_{a+1 a}, \quad |a - b| \geq 2, \\
\mathcal{S}_{a \pm 1}^{(+)} &= \mathcal{S}_{a \pm 1}^{(-)} = 0, \quad a \neq m, \\
\{E_{m-1 m+2}, E_{m m+1}\} &= \{E_{m+2 m-1}, E_{m+1 m}\} = 0,
\end{aligned} \tag{3}$$

where $q_a = q^{(-1)^{[a]}}$,

$$\begin{aligned}
\mathcal{S}_{a \pm 1}^{(+)} &= (E_{a a+1})^2 E_{a \pm 1 a+1 \pm 1} - (q + q^{-1}) E_{a a+1} E_{a \pm 1 a+1 \pm 1} E_{a a+1} \\
&\quad + E_{a \pm 1 a+1 \pm 1} (E_{a a+1})^2, \\
\mathcal{S}_{a \pm 1}^{(-)} &= (E_{a+1 a})^2 E_{a+1 \pm 1 a \pm 1} - (q + q^{-1}) E_{a+1 a} E_{a+1 \pm 1 a \pm 1} E_{a+1 a} \\
&\quad + E_{a+1 \pm 1 a \pm 1} (E_{a+1 a})^2,
\end{aligned}$$

and $E_{m-1 m+2}$ and $E_{m+2 m-1}$ are the $a = m - 1$, $b = m + 1$, cases of the following elements

$$\begin{aligned}
E_{ab} &= E_{ac} E_{cb} - q_c^{-1} E_{cb} E_{ac}, \\
E_{ba} &= E_{bc} E_{ca} - q_c E_{ca} E_{bc}, \quad a < c < b.
\end{aligned}$$

The \mathbb{Z}_2 grading of the algebra is specified such that the elements $K_a^{\pm 1}$, $\forall a \in \mathbf{I}$, and $E_{b b+1}$, $E_{b+1 b}$, $b \neq m$, are even, while $E_{m m+1}$ and $E_{m+1 m}$ are odd. Above, we have also used the notation $[a] = \begin{cases} 0, & \text{if } a \leq m, \\ 1, & \text{if } a > m. \end{cases}$

On the other hand, the Drinfeld version of $U_q(\mathfrak{g})$ is defined over $\mathbb{C}[[\hbar]]$, $q = \exp(\hbar)$, and is completed with respect to the \hbar - adic topology of $\mathbb{C}[[\hbar]]$. It is generated $\{E_{aa}, a \in \mathbf{I}; E_{b b+1}, E_{b+1 b}, b \in \mathbf{I}'\}$, subject to the same relations (3) with

$$K_a = q_a^{E_{aa}}.$$

It is well known that $U_q(\mathfrak{g})$ has the structure of a \mathbb{Z}_2 graded Hopf algebra, with a co - multiplication

$$\begin{aligned}
\Delta(E_{a a+1}) &= E_{a a+1} \otimes K_a K_{a+1}^{-1} + 1 \otimes E_{a a+1}, \\
\Delta(E_{a+1 a}) &= E_{a+1 a} \otimes 1 + K_a^{-1} K_{a+1} \otimes E_{a+1 a}, \\
\Delta(K_a^{\pm 1}) &= K_a^{\pm 1} \otimes K_a^{\pm 1},
\end{aligned}$$

co - unit

$$\begin{aligned}
\epsilon(E_{a a+1}) &= E_{a+1 a} = 0, \quad \forall a \in \mathbf{I}', \\
\epsilon(K_b^{\pm 1}) &= 1, \quad \forall b \in \mathbf{I},
\end{aligned}$$

and antipode

$$\begin{aligned} S(E_{aa+1}) &= -E_{aa+1}K_a^{-1}K_{a+1}, \\ S(E_{a+1a}) &= -K_aK_{a+1}^{-1}E_{a+1a}, \\ S(K_a^{\pm 1}) &= K_a^{\mp 1} \otimes K_a^{\mp 1}. \end{aligned}$$

At generic q , the Jimbo version of $U_q(\mathfrak{g})$ has more or less the same representation theory as that of the Drinfeld version [6]. Let $\{\epsilon_a | a \in \mathbf{I}\}$ be the basis of a vector space with a bilinear form $(\epsilon_a, \epsilon_b) = (-1)^{[a]}\delta_{ab}$. The roots of the classical Lie superalgebra $gl(m|n)$ can be expressed as

$$\epsilon_a - \epsilon_b, \quad a \neq b, \quad a, b \in \mathbf{I}.$$

For later use, we define

$$2\rho = \sum_{a \leq b} (-1)^{[a]+[b]} (\epsilon_a - \epsilon_b).$$

From [6] we know that every finite dimensional irreducible $U_q(\mathfrak{g})$ module is of highest weight type and is essentially uniquely characterized by a highest weight. Let $W(\lambda)$ be an irreducible $U_q(\mathfrak{g})$ module with highest weight $\lambda = \sum_a \lambda_a \epsilon_a$, $\lambda_a \in \mathbb{C}$. There exists a unique (up to scalar multiples) vector $v_+^\lambda \neq 0$ in $W(\lambda)$, called the highest weight vector, such that

$$\begin{aligned} E_{aa+1}v_+^\lambda &= 0, \quad a \in \mathbf{I}', \\ K_b v_+^\lambda &= q_b^{\lambda_b} v_+^\lambda, \quad b \in \mathbf{I}. \end{aligned}$$

$W(\lambda)$ is finite dimensional if and only if λ satisfies $\lambda_a - \lambda_{a+1} \in \mathbb{Z}_+$, $a \neq m$, and in that case, it has the same weight space decomposition as that of the corresponding irreducible $gl(m|n)$ module with the same highest weight.

2.3 Unitarity of covariant and contravariant tensor irreps

From this section on, we will assume that $U_q(\mathfrak{g})$ is obtained from the Jimbo algebra by specializing q to a *real positive parameter* different from 1. To construct a $*$ -operation for $U_q(\mathfrak{g})$, we first consider the Hopf subalgebra generated by $e = E_{aa+1}$, $f = E_{a+1a}$, and $k = K_a K_{a+1}^{-1}$, for a fixed $a \neq m$. It is not difficult to show that $*(e) = fk$, $*(f) = k^{-1}e$, $*(k^{\pm 1}) = k^{\pm 1}$ defines a $*$ -operation for this $U_q(sl(2))$ subalgebra. Possible generalizations of this to $U_q(\mathfrak{g})$ are

$$\begin{aligned} *(E_{aa+1}) &= (-1)^{(\theta+1)\delta_{ma}} E_{a+1a} K_a K_{a+1}^{-1}, \\ *(E_{a+1a}) &= (-1)^{(\theta+1)\delta_{ma}} K_a^{-1} K_{a+1} E_{aa+1}, \\ *(K_a^{\pm 1}) &= K_a^{\pm 1}, \end{aligned} \tag{4}$$

where $\theta = 1$ or 2 . It is quite obvious that the ‘quadratic’ relations of (3) are preserved by the $*$ -operations, and we have also explicitly checked that the ‘Serre relations’ are preserved as well. We will call the $*$ -operations type 1 and type 2 respectively when $\theta = 1$ and 2 .

It is also well known that

$$K_{2\rho} = \prod_{a < b} (K_a K_b^{-1})^{(-1)^{[a]+[b]}}$$

satisfies equation (2).

Now we consider the covariant and contravariant tensor irreps of $U_q(\mathfrak{g})$. The vector irrep π of $U_q(\mathfrak{g})$ is of highest weight ϵ_1 . The corresponding module \mathbb{E} has the standard basis $\{v_a | a \in \mathbf{I}\}$, such that

$$\begin{aligned} K_a v_b &= q_a^{\delta_{ab}} v_b, \\ E_{a \pm 1} v_b &= \delta_{b \pm 1} v_a. \end{aligned}$$

Define a sesquilinear form on $\mathbb{E} \otimes \mathbb{E}$ by

$$(v_a, v_b) = \delta_{ab} \prod_{c=1}^{a-1} q_c^{-1}.$$

Then it is straightforward to show that with respect to the type 1 $*$ - operation, we have

$$\begin{aligned} (E_{a \pm 1} v_b, v_c) &= (v_b, E_a^* v_c), \\ (K_a v_b, v_c) &= (v_b, K_a v_c). \end{aligned}$$

Therefore, the vector irrep is unitary of type 1.

The $U_q(\mathfrak{g})$ modules $\mathbb{E}^{\otimes k}$, $k \in \mathbb{Z}_+$ ($\mathbb{E}^0 = \mathbb{C}$), obtained by repeated tensor products of the vector module with itself can be decomposed into direct sums of irreducible type 1 unitary modules, and we will call each direct summand an irreducible contravariant tensor module, and the corresponding irreducible representation a contravariant tensor irrep.

The contravariant tensor irreps can be characterized in the following way. Let \mathbb{Z}_+ be the set of nonnegative integers. Define a subset \mathcal{P} of $\mathbb{Z}_+^{\otimes(m+n)}$ by

$$\mathcal{P} = \{p = (p_1, p_2, \dots, p_{m+n}) \in \mathbb{Z}_+^{\otimes(m+n)} \mid p_{m+1} \leq n, p_a \geq p_{a+1}, a \in \mathbf{I}'\}.$$

We associate with each $p \in \mathcal{P}$, a $\lambda^{(p)} = \sum_{a=1}^{m+n} \lambda_a \epsilon_a$ defined by

$$\begin{aligned} \lambda_a &= p_a, & a \leq m, \\ \sum_{\mu=1}^n \lambda_{m+\mu} \epsilon_{m+\mu} &= \sum_{\nu=1}^n \sum_{\mu=1}^{p_{m+\nu}} \epsilon_{m+\mu}. \end{aligned}$$

Introduce the set

$$\Lambda^{(1)} = \{\lambda^{(p)} \mid p \in \mathcal{P}\}. \quad (5)$$

From results of [15] [14] [6] we know that an irrep of $U_q(\mathfrak{g})$ is a contravariant tensor irrep if and only if its highest weight belongs to $\Lambda^{(1)}$. Needless to say, all such irreps are type 1 unitary.

Let $W(\lambda)$ be an irreducible contravariant tensor $U_q(\mathfrak{g})$ module with highest weight $\lambda \in \Lambda^{(1)}$. We define $\bar{\lambda}$ to be its lowest weight, and set $\lambda^\dagger = -\bar{\lambda}$. An explicit formula

for λ^\dagger was given in [14] (section III. B.), where a more compact characterization for the sets $\Lambda^{(1)}$ and $\Lambda^{(2)}$, which will be defined presently, was also given. We refer to that paper for details. Now the dual module $W(\lambda)^\dagger$ of $W(\lambda)$, which will we call a covariant tensor module, has highest weight λ^\dagger .

We introduce the set

$$\Lambda^{(2)} = \{\lambda^\dagger \mid \lambda \in \Lambda^{(1)}\}. \quad (6)$$

An irrep is a covariant tensor irrep if and only if its highest weight is contained in $\Lambda^{(2)}$, and all covariant tensor irreps are unitary of type 2. The most important example of type 2 unitary modules is the covariant vector module \mathbb{E}^\dagger , which is the dual of the vector module \mathbb{E} . Its highest weight is given by $-\epsilon_{m+n}$.

We summarize our discussions into the following

- Proposition 1** 1. *Each $U_q(\mathfrak{g})$ module $\mathbb{E}^{\otimes k}$ (resp. $(\mathbb{E}^\dagger)^{\otimes k}$), $k \in \mathbb{Z}_+$, can be decomposed into a direct sum of irreducible modules with highest weights belonging to $\Lambda^{(1)}$ (resp. $\Lambda^{(2)}$).*
2. *Every irreducible $U_q(\mathfrak{g})$ module with highest weight belonging to $\Lambda^{(1)}$ (resp. $\Lambda^{(2)}$) is contained in some repeated tensor products of \mathbb{E} (resp. \mathbb{E}^\dagger) as an irreducible component.*

More detailed structures of the covariant and contravariant tensor irreps can be understood, e.g., their characters and super characters can be computed, the Clebsch - Gordan problem of irreps within a given tensor type can also be resolved by using the supersymmetric Young diagram method. Here we elucidate some general aspects of the Clebsch - Gordan problem, which will play an important role in the remainder of the paper.

Denote by $[\lambda]$ the equivalence class of irreps with highest weight λ . For λ and λ' both belonging to $\Lambda^{(1)}$, we interpret $[\lambda] + [\lambda']$ as the equivalence class of the direct sum representations, and $[\lambda] \cdot [\lambda']$ as that of the direct products. Let $[\Lambda^{(1)}]$ be the \mathbb{Z}^+ module with a basis $\{[\lambda] \mid \lambda \in \Lambda^{(1)}\}$. Then the ‘ \cdot ’ operation defines a multiplication on $[\Lambda^{(1)}]$. Clearly $[\lambda] \cdot [0] = [0] \cdot [\lambda] = [\lambda]$. Further more, from section V of [14] we can deduce that if $[\lambda] \cdot [\lambda'] = [\lambda^1] + [\lambda^2] + \dots + [\lambda^k]$, then none of the λ^i is zero unless both λ and λ' are zero. This is in agreement with the fact that

$$\Lambda^{(1)} \cap \Lambda^{(2)} = \{0\}.$$

The discussions above can be repeated word by word for the irreps with highest weights belonging to $\Lambda^{(2)}$.

3 QUANTUM SPECIAL LINEAR SUPERGROUP $GL_q(m|n)$

For compact Lie groups in the classical setting, there exists the celebrated Tannaka - Krein duality theory [17], which enables the reconstruction of a group from the Hopf algebra of its representative functions. The theory of quantum groups [11] makes essential use of a quantum analog of the duality [16], and is formulated entirely in terms of the algebra of functions. We will adopt the same philosophy here to formulate and

study quantum supergroups. However, we should mention that Lie supergroups are much more complicated than ordinary compact Lie groups in structures; at the best, the Tannaka - Krein duality holds in a restricted sense for Lie supergroups even at the classical situation, though we have not come across any treatment of the problem in the literature.

3.1 Subalgebra of functions associated with the vector irrep

As before, we denote by π the vector irrep of $U_q(\mathfrak{g})$ relative to the standard basis $\{v_a \mid a \in \mathbf{I}\}$ of \mathbb{E} . Then

$$xv_a = \sum_b \pi(x)_{ba} v_b, \quad x \in U_q(\mathfrak{g}).$$

Let $(U_q(\mathfrak{g}))^0$ be the finite dual of $U_q(\mathfrak{g})$. Consider the elements t_{ab} , $a, b \in \mathbf{I}$ of $(U_q(\mathfrak{g}))^0$ satisfying

$$t_{ab}(x) = \pi(x)_{ab}, \quad \forall x \in U_q(\mathfrak{g}).$$

It is easy to show that the t_{ab} indeed belong to $(U_q(\mathfrak{g}))^0$. Also note that t_{ab} is even if $[a] + [b] \equiv 0 \pmod{2}$, and odd otherwise.

Standard Hopf algebra theory asserts that $(U_q(\mathfrak{g}))^0$ is a \mathbb{Z}_2 - graded Hopf algebra with its structures dualizing those of $U_q(\mathfrak{g})$. Consider the subalgebra G_q^π of $(U_q(\mathfrak{g}))^0$ generated by t_{ab} , $a, b \in \mathbf{I}$. The multiplication which G_q^π inherits from $(U_q(\mathfrak{g}))^0$ is given by

$$\begin{aligned} \langle t', x \rangle &= \sum_{(x)} \langle t \otimes t', x_{(1)} \otimes x_{(2)} \rangle \\ &= \sum_{(x)} (-1)^{[t'] [x_{(1)}]} \langle t, x_{(1)} \rangle \langle t', x_{(2)} \rangle, \quad \forall t, t' \in G_q^\pi, x \in U_q(\mathfrak{g}). \end{aligned} \quad (7)$$

To better understand the algebraic structure of G_q^π , we recall that the Drinfeld version of $U_q(\mathfrak{g})$ admits a universal R matrix, which in particular satisfies

$$R\Delta(x) = \Delta'(x)R, \quad \forall x \in U_q(\mathfrak{g}).$$

Applying $\pi \otimes \pi$ to both sides of the equation yields

$$R^{\pi\pi}(\pi \otimes \pi)\Delta(x) = (\pi \otimes \pi)\Delta'(x)R^{\pi\pi}, \quad (8)$$

where

$$\begin{aligned} R^{\pi\pi} &:= (\pi \otimes \pi)R \\ &= q^{\sum_{a \in \mathbf{I}} e_{aa} \otimes e_{aa} (-1)^{[a]}} + (q - q^{-1}) \sum_{a < b} e_{ab} \otimes e_{ba} (-1)^{[b]}. \end{aligned}$$

It is important to realize that equation (8) makes perfect sense within the Jimbo formulation of the quantized universal enveloping algebra $U_q(\mathfrak{g})$, even when q is specialized to a real parameter. We can re - interpret the equation in terms of the t_{ab} . Set $t = \sum_{a,b} e_{ab} \otimes t_{ab}$. Then

$$R_{12}^{\pi\pi} t_1 t_2 = t_2 t_1 R_{12}^{\pi\pi}. \quad (9)$$

The co - multiplication Δ of G_q^π is also defined in the standard way by

$$\langle \Delta(t_{ab}), x \otimes y \rangle = \langle t_{ab}, xy \rangle = \pi(xy)_{ab}, \quad \forall x, y \in U_q(\mathfrak{g}).$$

We have

$$\Delta(t_{ab}) = \sum_{c \in \mathbf{I}} (-1)^{([a]+[c])([c]+[b])} t_{ac} \otimes t_{cb}. \quad (10)$$

G_q^π also has the unit ϵ , and the co - unit $1_{U_q(\mathfrak{g})}$. Therefore, G_q^π has the structures of a \mathbb{Z}_2 - graded bi - algebra. However, it does not admit an antipode, as will be explained later.

Let $\pi^{(\lambda)}$ be an arbitrary irreducible contravariant tensor representation of $U_q(\mathfrak{g})$. We may also regard $\pi^{(\lambda)}$ as a representative of $[\lambda]$, where $\lambda \in \Lambda^{(1)}$. Define the elements $t_{ij}^{(\lambda)}$, $i, j = 1, 2, \dots, \dim_{\mathbb{C}} \pi^{(\lambda)}$, of $(U_q(\mathfrak{g}))^0$ by

$$t_{ij}^{(\lambda)}(x) = \pi^{(\lambda)}(x)_{ij}, \quad \forall x \in U_q(\mathfrak{g}).$$

It is an immediate consequence of Proposition 1 that $t_{ij}^{(\lambda)} \in G_q^\pi$, for all i, j and $\lambda \in \Lambda^{(1)}$, and every $f \in G_q^\pi$ can be expressed as a linear sum of these elements. From the representation theory of $U_q(\mathfrak{g})$ we can deduce that these elements are also linearly independent. Introduce the vector spaces

$$T^{(\lambda)} = \bigoplus_{i,j=1}^{\dim \pi^{(\lambda)}} \mathbb{C} t_{ij}^{(\lambda)}.$$

Then

Proposition 2 *As a vector space,*

$$G_q^\pi = \bigoplus_{\lambda \in \Lambda^{(1)}} T^{(\lambda)}.$$

Let us also denote the antipode of $(U_q(\mathfrak{g}))^0$ by S . Then for any $t_{ij}^{(\lambda)} \in G_q^\pi$,

$$S(t_{ij}^{(\lambda)})(x) = t_{ij}^{(\lambda)}(S(x)), \quad \forall x \in U_q(\mathfrak{g}).$$

That is, $S(t_{ij}^{(\lambda)})$ are the matrix elements of the dual irrep of $\pi^{(\lambda)}$, the highest weight of which is not contained in $\Lambda^{(1)}$ unless $\lambda = 0$. Therefore, G_q^π by itself does not admit an antipode.

3.2 Subalgebra of functions associated with the dual vector irrep

Let $\{\bar{v}_a \mid a \in \mathbf{I}\}$ be the basis of \mathbb{E}^\dagger dual to the standard basis of \mathbb{E} , i.e.,

$$\bar{v}_a(v_b) = \delta_{ab}.$$

Denote by $\bar{\pi}$ the covariant vector irrep relative to this basis. Let \bar{t}_{ab} , $a, b \in \mathbf{I}$, be the elements of $(U_q(\mathfrak{g}))^0$ such that

$$\bar{t}_{ab}(x) = \bar{\pi}(x)_{ab}, \quad \forall x \in U_q(\mathfrak{g}).$$

Note that \bar{t}_{ab} is even if $[a] + [b] \equiv 0 \pmod{2}$, and odd otherwise. These elements generate a \mathbb{Z}_2 - graded bi - subalgebra $G_q^{\bar{\pi}}$ of $(U_q(\mathfrak{g}))^0$ in the standard fashion. Here we merely point out that they obey the relation

$$R_{12}^{\bar{\pi}} \bar{t}_1 \bar{t}_2 = \bar{t}_2 \bar{t}_1 R_{12}^{\bar{\pi}}, \quad (11)$$

where

$$\begin{aligned} \bar{t} &= \sum_{a,b} e_{ab} \otimes \bar{t}_{ba}, \\ R^{\bar{\pi}} &= (\bar{\pi} \otimes \bar{\pi})R, \\ &= q^{\sum_{a \in \mathbf{I}} e_{aa} \otimes e_{aa} (-1)^{[a]}} + (q - q^{-1}) \sum_{a > b} e_{ab} \otimes e_{ba} (-1)^{[b]}. \end{aligned}$$

Also, the co - multiplication is given by

$$\Delta(\bar{t}_{ab}) = \sum_{c \in \mathbf{I}} (-1)^{([a] + [c])([c] + [b])} \bar{t}_{ac} \otimes \bar{t}_{cb}.$$

Denote by $\bar{\pi}^{(-\bar{\lambda})}$ the irrep dual to $\pi^{(\lambda)}$, $\lambda \in \Lambda^{(1)}$, in a given homogeneous basis. Introduce the elements $\bar{t}_{ij}^{(-\bar{\lambda})}$, $i, j = 1, 2, \dots, \dim_{\mathbb{C}} \pi^{(\lambda)}$, of $(U_q(\mathfrak{g}))^0$ such that

$$\bar{t}_{ij}^{(-\bar{\lambda})}(x) = \bar{\pi}_{ij}^{(-\bar{\lambda})}(x), \quad \forall x \in U_q(\mathfrak{g}).$$

Then it follows from Proposition 1 that these elements form a basis of $G_q^{\bar{\pi}}$. Set $\bar{T}^{(\mu)} = \oplus_{i,j} \mathbb{C} \bar{t}_{ij}^{(\mu)}$, we have

Proposition 3

$$G_q^{\bar{\pi}} = \bigoplus_{\mu \in \Lambda^{(2)}} \bar{T}^{(\mu)}.$$

3.3 Algebra G_q of functions on $GL_q(m|n)$

We define the algebra G_q of functions on the quantum general linear supergroup $GL_q(m|n)$ to be the \mathbb{Z}_2 - graded subalgebra of $(U_q(\mathfrak{g}))^0$ generated by $\{t_{ab}, \bar{t}_{ab} \mid a, b \in \mathbf{I}\}$. The t_{ab} and \bar{t}_{ab} , besides obeying the the relations (9) and (11), also satisfy

$$R_{12}^{\bar{\pi}} \bar{t}_1 t_2 = t_2 \bar{t}_1 R_{12}^{\bar{\pi}}, \quad (12)$$

where

$$\begin{aligned} R^{\bar{\pi}} &:= (\bar{\pi} \otimes \pi)R, \\ &= q^{-\sum_{a \in \mathbf{I}} e_{aa} \otimes e_{aa} (-1)^{[a]}} - (q - q^{-1}) \sum_{a < b} e_{ba} \otimes e_{ba} (-1)^{[a] + [b] + [a][b]}. \end{aligned}$$

Equation (12) enables us to factorize G_q into

$$G_q = G_q^{\pi} G_q^{\bar{\pi}}. \quad (13)$$

As both G_q^{π} and $G_q^{\bar{\pi}}$ are \mathbb{Z}_2 - graded bi - algebras, G_q inherits a natural bi - algebra structure. It also admits an antipode. By considering

$$(x \bar{v}_a)(v_b) = (-1)^{[x][a]} \bar{v}_a(S(x)v_b), \quad x \in U_q(\mathfrak{g}),$$

where $\{v_a\}$ is the standard basis of the vector irrep, and $\{\bar{v}_a\}$ is the basis of the covariant vector irrep dual to $\{v_a\}$, we arrive at

Lemma 1 *The antipode $S : G_q \rightarrow G_q$ is a linear anti - automorphism given by*

$$\begin{aligned} S(t_{ab}) &= (-1)^{[a][b]+[a]}\bar{t}_{ba}, \\ S(\bar{t}_{ab}) &= (-1)^{[a][b]+[b]}q^{(2\rho, \epsilon_a - \epsilon_b)}t_{ba}. \end{aligned} \quad (14)$$

Therefore, G_q has the structures of a \mathbb{Z}_2 - graded Hopf algebra.

Furthermore, $*$ - operations can also be constructed for G_q , thus turning it into a Hopf $*$ - superalgebra. We have

$$\begin{aligned} *(t_{ab}) &= (-1)^{(\theta+[a])([a]+[b])}\bar{t}_{ab}, \\ *(\bar{t}_{ab}) &= (-1)^{(\theta+[a])([a]+[b])}t_{ab}. \end{aligned}$$

where $\theta \in \mathbb{Z}_2$.

An important property of G_q is that it separates points of $U_q(\mathfrak{g})$, that is, for any nonvanishing $x \in U_q(\mathfrak{g})$, there exists $f \in G_q$ such that $f(x) \neq 0$. As a matter of fact, G_q^π by itself separates points of $U_q(\mathfrak{g})$. Put differently, for any $u \in U_q(\mathfrak{g})$, if $u \neq 0$, then $\pi^{\otimes p}(u) \neq 0$ for some $p \in \mathbb{Z}_+$.

To verify our assertion, we first consider the corresponding proposition in the classical situation of $U(\mathfrak{g})$ in detail. Let $E_{ab}^{(0)}$, $a, b \in \mathbf{I}$, be the standard generators of \mathfrak{g} embedded in its universal enveloping algebra. In the vector irrep $\pi^{(0)}$, one has

$$\pi^{(0)}(E_{ab}^{(0)}) = e_{ab}.$$

We isolate the $u(1)$ subalgebra of \mathfrak{g} with the generator

$$Z^{(0)} = \sum_{a \in \mathbf{I}} E_{aa}^{(0)},$$

and denote by $X_A^{(0)}$, $A = 1, \dots, (m+n)^2 - 1$, the elements $E_{cc}^{(0)} - E_{c+1, c+1}^{(0)}$, $c \in \mathbf{I}$, and $E_{ab}^{(0)}$, $a \neq b$, in any fixed ordering. Then a PBW basis for $U(\mathfrak{g})$ is given by,

$$\{B_{k, A_1 \dots A_l}^{(0)} = (Z^{(0)})^k X_{A_1}^{(0)} \dots X_{A_l}^{(0)} \mid k, l \in \mathbb{Z}_+, A_i \leq A_{i+1}, A_i \neq A_{i+1} \text{ if } [X_{A_i}^{(0)}] = 1\}.$$

Set $\pi^{(0)}(X_A^{(0)}) = e_A$. Denote by \mathcal{M} the vector space of $(m+n) \times (m+n)$ matrices, and define

$$\mathcal{R}_k = \sum_{i=0}^{k-1} \underbrace{\mathcal{M} \otimes \dots \otimes \mathcal{M}}_i \otimes I \otimes \underbrace{\mathcal{M} \otimes \dots \otimes \mathcal{M}}_{k-1-i}.$$

Let

$$b_{A_1 \dots A_k} = \sum_{\sigma \in S_k} (-1)^{|\sigma_{\{A\}}|} e_{A_{\sigma(1)}} \otimes e_{A_{\sigma(2)}} \otimes \dots \otimes e_{A_{\sigma(k)}},$$

where $|\sigma_{\{A\}}|$ is the number of permutations required amongst odd elements in order to change $X_{A_1} \otimes X_{A_2} \otimes \dots \otimes X_{A_k}$ to $X_{A_{\sigma(1)}} \otimes X_{A_{\sigma(2)}} \otimes \dots \otimes X_{A_{\sigma(k)}}$. Clearly, the elements

$$\{b_{A_1 \dots A_k} \mid k \in \mathbb{Z}_+, A_i \leq A_{i+1}, A_i \neq A_{i+1} \text{ if } [X_{A_i}] = 1\}$$

are linearly independent in $\mathcal{M}^{\otimes k}$, and we will denote by \mathcal{L}_k their linear span. By considering the trace (not the supertrace!) on each factor of $\mathcal{M}^{\otimes k}$, we can easily see that \mathcal{L}_k intersects \mathcal{R}_k trivially. Therefore,

$$\begin{aligned} (\pi^{(0)})^{\otimes(k+p)}(B_{0, A_1 \dots A_k}^{(0)}) &= \left((\pi^{(0)})^{\otimes k} \otimes (\pi^{(0)})^{\otimes p} \right) (B_{0, A_1 \dots A_k}^{(0)}) \\ &= b_{A_1 \dots A_k} \otimes I^{\otimes p} + r_{k,p}, \quad r_{k,p} \in \mathcal{R}_k \otimes \mathcal{M}^{\otimes p}, \end{aligned}$$

are linearly independent as elements of $\mathcal{M}^{\otimes(k+p)}$.

Consider $u \in U(\mathfrak{g})$ given by

$$u = \sum_{k=0}^K \sum_{l=0}^L \sum_{\{A\}} C_{k, A_1 \dots A_l} B_{k, A_1 \dots A_l}^{(0)}, \quad C_{k, A_1 \dots A_l} \in \mathbb{C}.$$

Using

$$(\pi^{(0)})^{\otimes p}(Z^k) = p^k I^{\otimes p},$$

we immediately see that $(\pi^{(0)})^{\otimes p}(u) = 0, \forall p > L$, requires

$$\sum_{k=0}^K p^k C_{k, A_1 \dots A_l} = 0, \quad \forall p > L,$$

which forces all the $C_{k, A_1 \dots A_l}$ to vanish. This completes the proof for the classical case.

Remark: There is something slightly unnatural about our proof, that is, the combination $E_{m m}^{(0)} - E_{m+1 m+1}^{(0)}$ does not belong to $sl(m|n) \subset \mathfrak{g}$, and this in turn forced us to consider the ordinary trace instead of the supertrace in proving $\mathcal{L}_k \cap \mathcal{R}_k = \{0\}$. We can avoid this unnaturalness when $m \neq n$ by using $E_{m m}^{(0)} + E_{m+1 m+1}^{(0)}$ instead, but not when $m = n$.

With the above preparations we can now readily prove our assertion for the quantum superalgebra. We first consider the Drinfeld version of $U_q(\mathfrak{g})$. Similar to the classical case, we set

$$Z = \sum_{a \in \mathbf{I}} E_{a a},$$

and denote by $X_A, A = 1, \dots, (m+n)^2 - 1$, the elements $E_{c c} - E_{c+1 c+1}, c \in \mathbf{I}'$ and $E_{a b}, a \neq b$, in a fixed ordering. Then

$$\{B_{k, A_1 \dots A_l} = Z^k X_{A_1} \dots X_{A_l} \mid k, l \in \mathbb{Z}_+, A_i \leq A_{i+1}, A_i \neq A_{i+1} \text{ if } [X_{A_i}] = 1\}$$

forms a PBW basis for $U_q(\mathfrak{g})$. Given

$$u = \hbar^k (u_0 + \hbar u_1 + \hbar^2 u_2 + \dots).$$

Each u_i is a finite \mathbb{C} - combination of some $B_{k, A_1 \dots A_l}$, and u_0 is assumed to be nonzero. Then it follows from the classical case that there exist infinitely many $p \in \mathbb{Z}_+$ such that

$$\pi^{\otimes p}(u) \not\equiv 0 \pmod{\hbar^{k+1}}.$$

For the Jimbo algebra, we observe that ordered monomials in E_{ab} , $a \neq b$, and $K_a^{\pm 1}$ form a basis of $U_q(\mathfrak{g})$. Given $u \in U_q(\mathfrak{g})$, and a positive integer p , we consider the matrix elements of $\pi^{\otimes p}(u)|_{q=\exp(\hbar)}$ as power series in \hbar . $\pi^{\otimes p}(u) \neq 0$ if and only if some of these power series do not vanish identically. Now for the purpose of computing $\pi^{\otimes p}(u)|_{q=\exp(\hbar)}$, we can make the identification

$$\begin{aligned}\pi^{\otimes p}(K_a) &= \sum_{k=0}^{\infty} \frac{(-1)^{k[a]} \hbar^k}{k!} e_{aa}(p)^k, \\ e_{aa}(p) &= \sum_{i=0}^{p-1} \underbrace{I \otimes \dots \otimes I}_i \otimes e_{aa} \otimes \underbrace{I \otimes \dots \otimes I}_{p-i-1}.\end{aligned}$$

This takes us back to the Drinfeld algebra situation, and we have already shown that in that situation the $\pi^{\otimes p}$, $p \in \mathbb{Z}_+$, separates points of $U_q(\mathfrak{g})$.

We summarize the discussions of this section into a Proposition, points ii) and iii) of which may be considered as a partial generalization of the classical Peter - Weyl theorem to the quantum supergroup in an algebraic setting:

Proposition 4 *i) G_q is a $*$ - Hopf superalgebra;
ii). G_q separates points of $U_q(\mathfrak{g})$;
iii). The following elements span G_q :*

$$\begin{aligned}t_{ij}^{(\lambda)} \bar{t}_{i'j'}^{(\mu)}, \quad & i, j = 1, 2, \dots, \dim \pi^{(\lambda)}, \quad \lambda \in \Lambda^{(1)}, \\ & i', j' = 1, 2, \dots, \dim \bar{\pi}^{(\mu)}, \quad \mu \in \Lambda^{(2)}.\end{aligned}$$

However, we should point out that these elements are *not* linearly independent.

4 INDUCED REPRESENTATIONS OF G_q

We will develop parabolic induction for representations of G_q in this section. Recall that corresponding to every locally finite right co - module $\omega : W \rightarrow W \otimes G_q$ over G_q , there exists a unique left $U_q(\mathfrak{g})$ module $U_q(\mathfrak{g}) \otimes W \rightarrow W$ with the module action defined by

$$xw = \omega(w)(x), \quad x \in U_q(\mathfrak{g}), \quad w \in W.$$

A similar correspondence exists for left G_q co - modules and right U_q modules. Therefore, we can describe the representation theory of G_q in both the G_q co - module language and $U_q(\mathfrak{g})$ module language, depending on which one is more convenient in a given situation. We will largely use the latter here.

4.1 Parabolic subalgebras of $U_q(gl(m|n))$

Let Θ be a subset of \mathbf{I}' . Introduce the following sets of elements of $U_q(\mathfrak{g})$:

$$\begin{aligned}\mathcal{S}_l &= \{K_a^{\pm 1}, a \in \mathbf{I}; E_{c \ c+1}, E_{c+1 \ c}, c \in \Theta\}; \\ \mathcal{S}_{p+} &= \mathcal{S}_l \cup \{E_{c \ c+1}, c \in \mathbf{I}' \setminus \Theta\}; \\ \mathcal{S}_{p-} &= \mathcal{S}_l \cup \{E_{c+1 \ c}, c \in \mathbf{I}' \setminus \Theta\}.\end{aligned}$$

The elements of each set generate a \mathbb{Z}_2 - graded Hopf subalgebra of $U_q(\mathfrak{g})$. We denote by $U_q(\mathfrak{l})$ the Hopf subalgebra generated by the elements of \mathcal{S}_l , and by $U_q(\mathfrak{p}_\pm)$ the Hopf subalgebras respectively generated by the elements of \mathcal{S}_{p_\pm} . In the classical limit, the Hopf subalgebras $U_q(\mathfrak{p}_\pm)$ coincide with the universal enveloping algebras of parabolic subalgebras of the Lie superalgebra \mathfrak{g} . Therefore, we will call $U_q(\mathfrak{p}_\pm)$ parabolic subalgebras of $U_q(\mathfrak{g})$.

Let V_μ be a finite dimensional irreducible $U_q(\mathfrak{l})$ module. Then V_μ is of highest weight type. Let μ be the highest weight and $\tilde{\mu}$ the lowest weight of V_μ respectively. We can extend V_μ in a unique fashion to a $U_q(\mathfrak{p}_+)$ module, which we still denote by V_μ , such that the elements of $\mathcal{S}_{p_+} \setminus \mathcal{S}_l$ act by zero. Similarly, V_μ also leads to a $U_q(\mathfrak{p}_-)$ module, on which the elements of $\mathcal{S}_{p_-} \setminus \mathcal{S}_l$ act by zero. It is not difficult to see that all finite dimensional irreducible $U_q(\mathfrak{p}_\pm)$ modules are of this kind.

Consider a finite dimensional irreducible $U_q(\mathfrak{g})$ module $W(\lambda)$ with highest weight λ and lowest weight $\bar{\lambda}$. $W(\lambda)$ can be restricted into a $U_q(\mathfrak{p}_+)$ or $U_q(\mathfrak{p}_-)$ module in a natural way, and the resultant module is always indecomposable, but not irreducible in general.

Consider first the case of $U_q(\mathfrak{p}_+)$. We wish to examine the \mathbb{Z}_2 - graded vector space $Hom_{U_q(\mathfrak{p}_+)}(W(\lambda), V_\mu)$, which graded - commutes with $U_q(\mathfrak{p}_+)$, namely,

$$p\phi - (-1)^{[p][\phi]}\phi p = 0, \quad p \in U_q(\mathfrak{p}_+), \quad \phi \in Hom_{U_q(\mathfrak{p}_+)}(W(\lambda), V_\mu).$$

Because of the irreducibility of V_μ , every non - zero $\phi \in Hom_{U_q(\mathfrak{p}_+)}(W(\lambda), V_\mu)$ must be surjective, and thus $V_\mu \cong W(\lambda)/Ker\phi$. As a $U_q(\mathfrak{p}_+)$ module, $W(\lambda)$ is indecomposable, and contains a unique maximal proper submodule M such that the lowest weight vector w_- of $W(\lambda)$ does not belong to M . Therefore, $Ker\phi = M$, and $V_\mu = \phi(U_q(\mathfrak{l})w_-)$. This forces $\bar{\lambda} = \tilde{\mu}$, and all elements of $Hom_{U_q(\mathfrak{p}_+)}(W(\lambda), V_\mu)$ are scalar multiples of one another. It is worth observing that the map ϕ may be odd. In fact its degree is given by $[\phi] \equiv [w_-] + [\phi(w_-)] \pmod{2}$. The case of $U_q(\mathfrak{p}_-)$ can be studied in exactly the same way. To summarize, we have

Lemma 2

$$\begin{aligned} \dim_{\mathbb{C}} Hom_{U_q(\mathfrak{p}_+)}(W(\lambda), V_\mu) &= \begin{cases} 1, & \bar{\lambda} = \tilde{\mu}, \\ 0, & \bar{\lambda} \neq \tilde{\mu}. \end{cases} \\ \dim_{\mathbb{C}} Hom_{U_q(\mathfrak{p}_-)}(W(\lambda), V_\mu) &= \begin{cases} 1, & \lambda = \mu, \\ 0, & \lambda \neq \mu. \end{cases} \end{aligned}$$

4.2 Induced representations and quantum superbundles

Let us first introduce two types of left actions of $U_q(\mathfrak{g})$ on G_q , which correspond to the left and right translations in the classical situation.

Define a bilinear map $\cdot : U_q(\mathfrak{g}) \otimes G_q \rightarrow G_q$ by

$$\begin{aligned} x \otimes f &\mapsto x \cdot f \\ &= \sum_{(f)} \langle f_{(1)}, S^{-1}(x) \rangle f_{(2)}, \end{aligned} \tag{15}$$

which can be easily shown to satisfy

$$\begin{aligned} (x \cdot f)(y) &= (-1)^{[x][y]} f(S^{-1}(x)y), \\ x \cdot (y \cdot f) &= (xy) \cdot f, \quad x, y \in U_q(\mathfrak{g}), \quad f \in G_q. \end{aligned}$$

(We assume that the elements $x, y \in U_q(\mathfrak{g})$ and $g, f \in G_q$ are homogeneous for the sake of simplicity. All the statements below generalize to inhomogeneous elements in the obvious way.) Therefore, this defines a left action of $U_q(\mathfrak{g})$ on G_q , which corresponds to the left translation of Lie groups in the classical situation. It is worth observing that we may replace S^{-1} in the above definition, and arrive at a different left action.

Another left action ‘ \circ ’ of $U_q(\mathfrak{g})$ on G_q can be defined by

$$x \circ f = \sum_{(f)} f_{(1)} (-1)^{[x]([f]+[x])} \langle f_{(2)}, x \rangle. \quad (16)$$

Straightforward calculations can show that

$$\begin{aligned} y \circ (x \circ f) &= (xy) \circ f; \\ (x \circ f)(y) &= f(yx), \\ (id_{G_q} \otimes x \circ) \Delta(f) &= \Delta(x \circ f). \end{aligned}$$

This corresponds to the right translation in the classical theory. It graded - commutes with the action ‘ \cdot ’, namely,

$$x \circ (y \cdot f) = (-1)^{[x][y]} y \cdot (x \circ f).$$

Let $U_q(\mathfrak{p})$ denote either $U_q(\mathfrak{p}_+)$ or $U_q(\mathfrak{p}_-)$. Given any finite dimensional left $U_q(\mathfrak{p})$ module V , we form the tensor product $V \otimes_{\mathbb{C}} G_q$, which is a subspace of functions $U_q(\mathfrak{g}) \rightarrow V$:

$$\begin{aligned} \zeta &= \sum v_i \otimes f_i \in V \otimes G_q, \\ x &\in U_q(\mathfrak{g}), \\ \zeta(x) &= \sum f_i(x) v_i. \end{aligned}$$

The left actions ‘ \cdot ’ and ‘ \circ ’ of $U_q(\mathfrak{g})$ on G_q can be extended in an obvious way to actions on $V \otimes_{\mathbb{C}} G_q$

$$\begin{aligned} x \cdot \zeta &= \sum (-1)^{[x][v_i]} v_i \otimes x \cdot f_i, \\ x \circ \zeta &= \sum (-1)^{[x][v_i]} v_i \otimes x \circ f_i, \quad x \in U_q(\mathfrak{g}). \end{aligned}$$

Furthermore, there also exists a co - action ω of G_q on $V \otimes_{\mathbb{C}} G_q$ defined by $\omega = id_V \otimes \Delta'$, where Δ' represents the opposite co - multiplication of G_q .

Consider the subspace of $V \otimes_{\mathbb{C}} G_q$ defined by

$$\mathcal{O}^V = \{ \zeta \in V \otimes_{\mathbb{C}} G_q \mid p \circ \zeta = (S(p) \otimes id_{G_q}) \zeta, \forall p \in U_q(\mathfrak{p}) \}.$$

Lemma 3 \mathcal{O}^V furnishes a left $U_q(\mathfrak{g})$ module under ‘ \cdot ’, and at the same time a right G_q co - module under ω .

Proof: The Lemma can be confirmed by direct calculations. For $x \in U_q(\mathfrak{g})$, $p \in U_q(\mathfrak{p})$, $\zeta \in \mathcal{O}^V$, we have

$$\begin{aligned}
p \circ (x \cdot \zeta) &= (-1)^{[x][p]} x \cdot (p \circ \zeta) \\
&= (S(p) \otimes id_{G_q})(x \cdot \zeta); \\
(p \circ id_{G_q})\omega(\zeta) &= (p \circ id_{G_q})(id_V \otimes \Delta')\zeta \\
&= (id_V \otimes \tau)(id_V \otimes id_{G_q} \otimes p \circ)(id_V \otimes \Delta)\zeta \\
&= (id_V \otimes \tau)(id_V \otimes \Delta')(p \circ \zeta) \\
&= \omega(S(p) \otimes id_{G_q})\zeta,
\end{aligned}$$

where τ is the flip mapping.

We call \mathcal{O}^V the induced $U_q(\mathfrak{g})$ module, and also the induced G_q co - module, which gives rise to a representation of G_q . A conceptual understanding of \mathcal{O}^V can be gained by considering its classical analog. Let P be a parabolic subgroup of the complex Lie supergroup $SL(m|n)$, and E a finite dimensional representation of P . Then $SL(m|n) \times_P E$, the quotient space of $SL(m|n) \times E$ under the equivalence relation $(g, v) \sim (gp, p^{-1}v)$ for all $p \in P$, defines a super vector bundle over the supermanifold $SL(m|n)/P$. A function $f : SL(m|n) \rightarrow E$ satisfying $f(gp) = p^{-1}f(g)$, $\forall p \in P$ defines a section of the bundle $s_f : SL(m|n)/P \rightarrow SL(m|n) \times_P E$. Analogously, we may regard \mathcal{O}^V as the vector space of sections of a quantum super vector bundle over the quantum counter part of $SL(m|n)/P$.

It is of great importance to systematically develop the theory of quantum homogeneous super vector bundles, and the subject will be investigated to depth in a forthcoming publication. In this paper, we will restrict ourselves to issues directly related to representation theory, and will not further ponder on noncommutative geometry, except for the last section, where we will discuss in some detail quantum projective superspaces when dealing with explicit realizations of the skew supersymmetric tensor irreps and their duals.

We have the following quantum analog of Frobenius reciprocity.

Proposition 5 *Let W be a quotient $U_q(\mathfrak{g})$ module of $\bigoplus_{k,l=0}^{\infty} \mathbb{E}^{\otimes k} \otimes (\mathbb{E}^*)^{\otimes l}$ (the restriction of which furnishes a $U_q(\mathfrak{p})$ module in a natural way.). Then there is a canonical isomorphism*

$$Hom_{U_q(\mathfrak{g})}(W, \mathcal{O}^V) \cong Hom_{U_q(\mathfrak{p})}(W, V), \quad (17)$$

Proof: We prove the Proposition by explicitly constructing the isomorphism, which we claim to be the linear map

$$\begin{aligned}
F : Hom_{U_q(\mathfrak{g})}(W, \mathcal{O}^V) &\rightarrow Hom_{U_q(\mathfrak{p})}(W, V), \\
\psi &\mapsto \psi(1_{U_q(\mathfrak{g})}),
\end{aligned}$$

with the inverse map

$$\begin{aligned}
\bar{F} : Hom_{U_q(\mathfrak{p})}(W, V) &\rightarrow Hom_{U_q(\mathfrak{g})}(W, \mathcal{O}^V), \\
\phi &\mapsto \bar{\phi},
\end{aligned}$$

where $\bar{\phi}$ is defined by

$$\bar{\phi}(w)(x) = (-1)^{[x]([w]+1)} \phi(S(x)w), \quad x \in U_q(\mathfrak{g}), \quad w \in W.$$

As for F , we need to show that its image is contained in $\text{Hom}_{U_q(\mathfrak{p})}(W, V)$. This is indeed the case, as

$$\begin{aligned} p(F\psi(w)) &= (p \cdot \psi(w))(1_{U_q(\mathfrak{g})}) \\ &= (-1)^{[\psi][p]} F\psi(pw), \quad p \in U_q(\mathfrak{p}), \quad w \in W. \end{aligned}$$

In order to show that \bar{F} is the inverse of F , we first need to demonstrate that the image $\text{Im}(\bar{F})$ of \bar{F} is contained in $\text{Hom}_{U_q(\mathfrak{g})}(W, \mathcal{O}^V)$. Note that $\text{Im}(\bar{F}) \subset \text{Hom}_{\mathbb{C}}(W, V \otimes G_q)$, since W is a subquotient of $\bigoplus_{k,l=0}^{\infty} \mathbb{E}^{\otimes k} \otimes (\mathbb{E}^*)^{\otimes l}$. Some relatively simple manipulations lead to

$$\begin{aligned} (y \cdot \bar{\phi}(w))(x) &= (-1)^{[y][\bar{\phi}] + [x]([w] + [x] + [y])} \phi(S(x)yw) \\ &= (-1)^{[y][\bar{\phi}]} \bar{\phi}(yw)(x), \\ (p \circ \bar{\phi}(w))(x) &= (-1)^{[x]([w] + 1) + [p][\phi]} \phi(S(p)S(x)w) \\ &= S(p)(\bar{\phi}(w)(x)), \quad x, y \in U_q(\mathfrak{g}), \quad p \in U_q(\mathfrak{p}), \quad w \in W. \end{aligned}$$

Therefore, $\text{Im}(\bar{F}) \subset \text{Hom}_{U_q(\mathfrak{g})}(W, \mathcal{O}^V)$.

Now we show that F and \bar{F} are inverse to each other. For $\psi \in \text{Hom}_{U_q(\mathfrak{g})}(W, \mathcal{O}^V)$, and $\phi \in \text{Hom}_{U_q(\mathfrak{p})}(W, V)$, we have

$$\begin{aligned} (F\bar{F}\phi)(w) &= (\bar{F}\phi)(w)(1_{U_q(\mathfrak{g})}) \\ &= \phi(w), \\ (\bar{F}F\psi)(w)(x) &= (-1)^{[x]([w] + 1)} (F\psi)(S(x)w) \\ &= (-1)^{[x]([w] + 1)} \psi(S(x)w)(1_{U_q(\mathfrak{g})}) \\ &= (-1)^{[x]([\psi(w)] + 1)} (S(x) \cdot \psi(w))(1_{U_q(\mathfrak{g})}) \\ &= \psi(w)(x), \quad x \in U_q(\mathfrak{g}), \quad w \in W. \end{aligned}$$

This completes the proof of the Proposition.

4.3 Quantum Borel - Weil theorem for covariant and contravariant tensor irreps

In this subsection we study in detail the covariant and contravariant tensor irreps of $U_q(\mathfrak{g})$ within the framework of parabolic induction. Our main result here will be a quantum version of the Borel - Weil theorem for these irreps.

Let V be a finite dimensional irreducible $U_q(\mathfrak{p})$ module, with the $U_q(\mathfrak{l})$ highest weight μ and $U_q(\mathfrak{l})$ lowest weight $\tilde{\mu}$. For the purpose of studying the tensor representations, we only need to consider

$$\begin{aligned} \overline{\mathcal{O}}(\mu) &= \mathcal{O}^V \cap (V \otimes G_q^\pi), \\ \mathcal{O}(\mu) &= \mathcal{O}^V \cap (V \otimes G_q^{\bar{\pi}}). \end{aligned}$$

Let us study $\mathcal{O}(\mu)$ first. A typical element of $\mathcal{O}(\mu)$ is of the form

$$\zeta = \sum_{\lambda \in \Lambda^{(1)}} \sum_{\alpha, \beta, i} c_{\alpha\beta, i}^\lambda v_i \otimes \bar{t}_{\alpha\beta}^{(\lambda^\dagger)},$$

where $\{v_i\}$ is a basis of V , and the $c_{\alpha\beta, i}^\lambda$ are complex numbers. The $\bar{t}_{\alpha\beta}^{(\lambda^\dagger)}$ are elements of the Peter - Weyl basis for G_q^π , which, needless to say, are polynomials in \bar{t}_{ab} , $a, b \in \mathbf{I}$. The property that $(p \circ \zeta) = (S(p) \otimes id_{G_q})\zeta$, $\forall p \in U_q(\mathfrak{p})$ leads to

$$\sum_{\gamma, i} (-1)^{[p]([\gamma] + [v_i])} c_{\alpha\gamma, i}^\lambda t_{\gamma\beta}^{(\lambda)}(p) v_i = \sum_i c_{\alpha\beta, i}^\lambda p v_i, \quad \forall p \in U_q(\mathfrak{p}). \quad (18)$$

Let $W(\lambda)$ with the basis $\{w_\alpha\}$ be the irreducible $U_q(\mathfrak{g})$ module associated with the irrep $t^{(\lambda)}$. We define the linear maps between \mathbb{Z}_2 graded *vector spaces*

$$\begin{aligned} \phi_\lambda^{(\alpha)} : W(\lambda) &\rightarrow V, \\ w_\beta &\mapsto \sum_i c_{\alpha\beta, i}^\lambda v_i. \end{aligned}$$

There is no particular significance attached to the maps at this stage, apart from the mere fact that they can be employed to re - express equation (18) as

$$\sum_\gamma (-1)^{[p][\phi_\lambda^{(\alpha)}]} t_{\gamma\beta}^{(\lambda)}(p) \phi_\lambda^{(\alpha)}(w_\gamma) = p \phi_\lambda^{(\alpha)}(w_\beta).$$

We emphasize that this equation is entirely equivalent to (18). Now something of crucial importance appears: this equation requires that each $\phi_\lambda^{(\alpha)}$ be a $U_q(\mathfrak{p})$ module homomorphism of degree $[\phi_\lambda^{(\alpha)}]$. Lemma 2 forces

$$\phi_\lambda^{(\alpha)} = c_\alpha \phi_\lambda, \quad c_\alpha \in \mathbb{C},$$

and ϕ_λ may be nonzero only when

$$\begin{aligned} i). \quad \bar{\lambda} &= \tilde{\mu}, \quad \text{if } U_q(\mathfrak{p}) = U_q(\mathfrak{p}_+), \\ ii). \quad \lambda &= \mu, \quad \text{if } U_q(\mathfrak{p}) = U_q(\mathfrak{p}_-). \end{aligned}$$

In these cases, $\mathcal{O}(\mu)$ is spanned by

$$\zeta_\alpha = \sum_\beta \phi_\lambda(w_\beta) \otimes \bar{t}_{\alpha\beta}^{(\lambda^\dagger)},$$

which are obviously linearly independent. Furthermore,

$$x \cdot \zeta_\alpha = (-1)^{[x][\phi_\lambda]} \sum_\beta t_{\beta\alpha}^{(\lambda)}(x) \zeta_\beta, \quad x \in U_q(\mathfrak{g}). \quad (19)$$

The case of $\overline{\mathcal{O}}(\mu)$ can be studied in exactly the same way. To summarize, we have the following quantum analog of Borel - Weil theorem for the covariant and contravariant tensor irreps

Proposition 6 *As $U_q(\mathfrak{g})$ modules,*

$$\mathcal{O}(\mu) \cong \begin{cases} W((- \tilde{\mu})^\dagger), & \text{if } \tilde{\mu} \in -\Lambda^{(2)}, \quad U_q(\mathfrak{p}) = U_q(\mathfrak{p}_+), \\ W(\mu), & \text{if } \mu \in \Lambda^{(1)}, \quad U_q(\mathfrak{p}) = U_q(\mathfrak{p}_-), \\ \{0\}, & \text{otherwise.} \end{cases} \quad (20)$$

$$\overline{\mathcal{O}}(\mu) \cong \begin{cases} W((- \tilde{\mu})^\dagger), & \text{if } \tilde{\mu} \in -\Lambda^{(1)}, \quad U_q(\mathfrak{p}) = U_q(\mathfrak{p}_+), \\ W(\mu), & \text{if } \mu \in \Lambda^{(2)}, \quad U_q(\mathfrak{p}) = U_q(\mathfrak{p}_-), \\ \{0\}, & \text{otherwise.} \end{cases} \quad (21)$$

In the Proposition, the notation $W(\lambda)$ signifies the irreducible $U_q(\mathfrak{g})$ module with highest weight λ .

5 QUANTUM PROJECTIVE SUPERSPACES, SKEW SUPERSYMMETRIC TENSOR IRREPS AND THEIR DUALS

We will apply the general theory developed in the last section to study two infinite classes of irreps, namely, the skew supersymmetric tensor irreps and their dual irreps. Explicit realizations of these irreps will be given in terms of sections of quantum super vector bundles over quantum projective superspaces.

5.1 Quantum projective superspaces

Let $U_q(\mathfrak{g}')$, $\mathfrak{g}' = gl(m|n-1)$, be the subalgebra of $U_q(\mathfrak{g})$ generated by the following elements

$$\{K_a, a \in \mathbf{I}'; E_{c+1}, E_{c+1}^{-1}, c \in \mathbf{I}' \setminus \{m+n-1\}\}.$$

Clearly $U_q(\mathfrak{g}')$ is a Hopf subalgebra. Define

$$\begin{aligned} \mathcal{A}_+ &= \{f \in G_q^\pi \mid f(xp) = \epsilon(p)f(x), \forall x \in U_q(\mathfrak{g}), p \in U_q(\mathfrak{g}')\}, \\ \mathcal{A}_- &= \{f \in G_q^\pi \mid f(xp) = \epsilon(p)f(x), \forall x \in U_q(\mathfrak{g}), p \in U_q(\mathfrak{g}')\}. \end{aligned}$$

The Hopf algebra structure of $U_q(\mathfrak{g}')$ implies that both \mathcal{A}_+ and \mathcal{A}_- are subalgebras of G_q . Together they generate another subalgebra of G_q , which we will denote by $\mathbb{S}_q^{m|n-1}$. Set

$$z_a = t_{a \ m+n}, \quad \bar{z}_a = \bar{t}_{a \ m+n}, \quad a \in \mathbf{I}.$$

Then z_a and \bar{z}_a are conjugate to each other under the $*$ -operation with $\theta = 0$. More explicitly,

$$*(z_a) = \bar{z}_a, \quad \forall a \in \mathbf{I}.$$

Now $\mathbb{S}_q^{m|n-1}$ is generated by the z 's and \bar{z} 's, which satisfy the following commutations relations

$$\begin{aligned} z_a z_b &= (-1)^{[z_a][z_b]} q z_b z_a, \quad a < b, \\ (z_c)^2 &= 0, \quad c \leq m; \\ \bar{z}_a \bar{z}_b &= (-1)^{[\bar{z}_a][\bar{z}_b]} q^{-1} \bar{z}_b \bar{z}_a, \quad a < b, \\ (\bar{z}_c)^2 &= 0, \quad c \leq m; \\ \bar{z}_a z_b &= q(-1)^{[\bar{z}_a][z_b]} z_b \bar{z}_a + \delta_{ab} \left\{ (1 - q^{-1}) \bar{z}_a z_a \right. \\ &\quad \left. - (-1)^{[\bar{z}_a]} (q - q^{-1}) \sum_{c < a} \bar{z}_c z_c \right\}, \quad \forall a, b \in \mathbf{I}, \\ \sum_{c \in \mathbf{I}} \bar{z}_c z_c &= 1. \end{aligned}$$

It can be shown that the last two equations imply that

$$\sum_{c \in \mathbf{I}} q^{(2\rho, \epsilon_c)} z_c \bar{z}_c = q^{(2\rho, \epsilon_{m+n})}.$$

$\mathbb{S}_q^{m|n-1}$ furnishes a right G_q co - module algebra, with the co - module action $\omega : \mathbb{S}_q^{m|n-1} \rightarrow \mathbb{S}_q^{m|n-1} \otimes G_q$ defined by

$$\begin{aligned}\omega(z_a) &= \sum_{c \in \mathbf{I}} z_c \otimes t_{ac}, \\ \omega(\bar{z}_a) &= \sum_{c \in \mathbf{I}} \bar{z}_c \otimes \bar{t}_{ac}.\end{aligned}$$

Also, $\mathbb{S}_q^{m|n-1}$ gives rise to a right $U_q(\mathfrak{g})$ module algebra with the module action ‘ \circ ’ defined by (16). This module algebra structure restricts naturally to a module algebra structure over $U_q(\mathfrak{g}') \otimes U_q(gl(1))$, where $U_q(gl(1))$ is generated by $K_{m+n}^{\pm 1}$. The action of $U_q(\mathfrak{g}')$ on $\mathbb{S}_q^{m|n-1}$ is trivial as following the definitions of \mathcal{A}_{\pm} ; $U_q(gl(1))$ also acts in a very simple manner. To be explicit, we introduce the notations that for $L = (\theta_1, \dots, \theta_m; l_1, \dots, l_n) \in \{0, 1\}^{\otimes m} \otimes \mathbb{Z}_+^{\otimes n}$, $|L| = \sum_{i=1}^m \theta_i + \sum_{\mu=1}^n l_{\mu}$. Set

$$\begin{aligned}Z^L &= z_1^{\theta_1} \dots z_m^{\theta_m} z_{m+1}^{l_1} \dots z_{m+n}^{l_n}, \\ \bar{Z}^L &= \bar{z}_1^{\theta_1} \dots \bar{z}_m^{\theta_m} \bar{z}_{m+1}^{l_1} \dots \bar{z}_{m+n}^{l_n}.\end{aligned}\tag{22}$$

Then for any $k \in \mathbb{Z}$, and $p \in U_q(\mathfrak{g}')$, we have

$$(pK_{m+n}^k) \circ (Z^L \bar{Z}^{L'}) = \epsilon(p)q^{k(|L'| - |L|)} Z^L \bar{Z}^{L'}.\tag{23}$$

We will define the quantum projective superspace $\mathbb{C}P_q^{m|n-1}$ to be the $U_q(gl(1))$ invariant subalgebra of $\mathbb{S}_q^{m|n-1}$, namely,

$$\mathbb{C}P_q^{m|n-1} = \left(\mathbb{S}_q^{m|n-1} \right)^{U_q(gl(1))}.\tag{24}$$

5.2 Skew supersymmetric tensor irreps and their duals

We specialize $U_q(\mathfrak{p}_+)$ and $U_q(\mathfrak{p}_-)$ to the case with $\Theta = \mathbf{I}' \setminus \{m+n-1\}$. Consider a one - dimensional irreducible $U_q(\mathfrak{p}_+)$ module $V_+ = \mathbb{C}v$ such that

$$\begin{aligned}E_{bb+1}v &= E_{c+1}cv = 0, \\ K_bv &= v, \\ K_{m+n}v &= q^{-k}v, \\ k \in \mathbb{Z}_+, \quad b, c \in \mathbf{I}', \quad c < m+n-1,\end{aligned}$$

and denote the associated representation by ϕ . Define

$$\bar{\mathcal{O}}_k = \left\{ \zeta \in V_+ \otimes G_q^\pi \mid (p \circ \zeta)(x) = \phi(S(p))\zeta(x), \forall x \in U_q(\mathfrak{g}), p \in U_q(\mathfrak{p}_+) \right\}.$$

Direct calculations can show that

$$\bar{\mathcal{O}}_k = \bigoplus_{|L|=k} \mathbb{C}v \otimes Z^L,\tag{25}$$

where Z^L is defined by (22). Then $\bar{\mathcal{O}}_k$ gives rise to the rank k skew supersymmetric tensor irrep of $U_q(\mathfrak{g})$, with the highest weight

$$\lambda = \begin{cases} \sum_{i=1}^k \epsilon_i, & k \leq m, \\ \sum_{i=1}^m \epsilon_i + (k-m)\epsilon_{m+1}, & k > m. \end{cases}$$

Now let $V_- = \mathbb{C}w$ be a one dimensional irreducible $U_q(\mathfrak{p}_-)$ module such that

$$\begin{aligned} E_{c+1}v &= E_{b+1}bv = 0, \\ K_bv &= v, \\ K_{m+n}v &= q^k v, \\ k \in \mathbb{Z}_+, \quad b, c \in \mathbf{I}', \quad c < m+n-1, \end{aligned}$$

and denote the corresponding irrep by ψ . Define

$$\mathcal{O}_k = \left\{ \zeta \in V_- \otimes G_q^{\bar{\pi}} \mid (p \circ \zeta)(x) = \psi(S(p))\zeta(x), \forall x \in U_q(\mathfrak{g}), p \in U_q(\mathfrak{p}_-) \right\}.$$

Then

$$\mathcal{O}_k = \bigoplus_{|L|=k} \mathbb{C}w \otimes \overline{Z}^L. \quad (26)$$

This time \mathcal{O}_k yields an irrep with highest weight

$$\lambda = -k\epsilon_{m+n},$$

which is dual to the rank k skew supersymmetric tensor irrep.

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